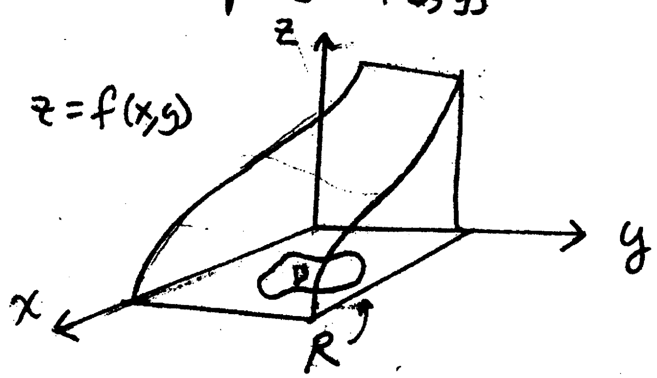
DEFINING INTEGRALS OVER MORE GENERAL REGIONS (Domains D)

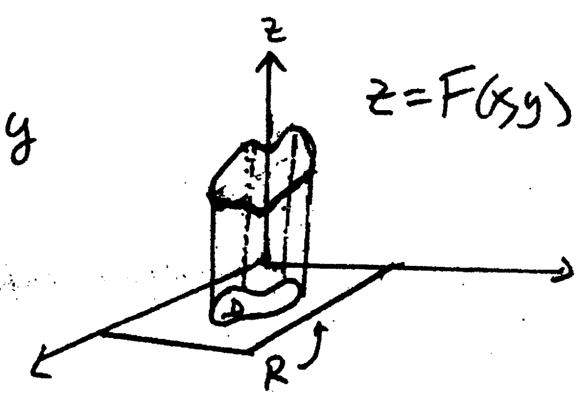
Given function $z = f(x,y)$.

Suppose D is a bounded region of the x-y plane.
("bounded" means that it lies completely inside some rectangle R)
Let R be a rectangle that contains D.
Define $F(x,y)$ as follows: $F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$

GRAPH of $z = f(x,y)$

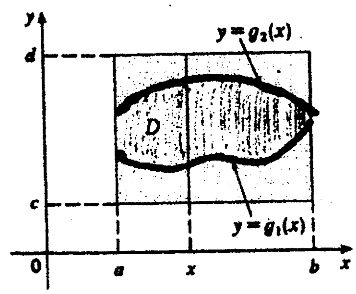


GRAPH of $z = F(x,y)$



DEFINE $\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$

Type I REGIONS



$a \le x \le b$

$g_1(x) \le y \le g_2(x)$

FIGURE 6

SEC 16.3

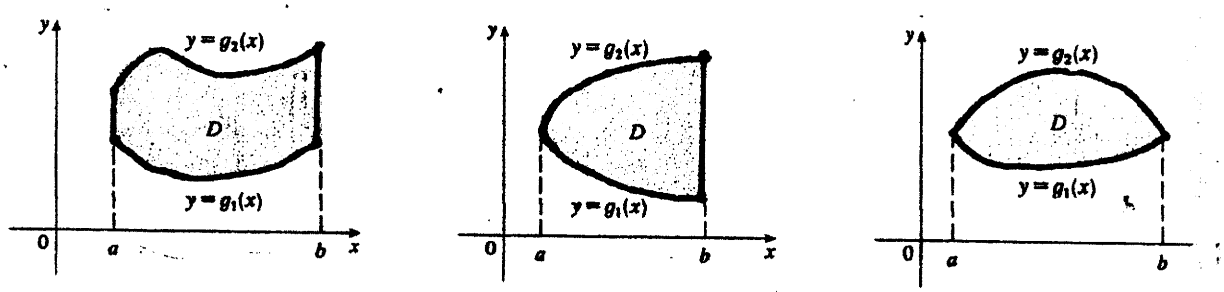


FIGURE 5 Some type I regions

3 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Type I Region Example

D is the region bounded by the graphs of
$y = 2x^2$ and $y = 1 + x^2$.

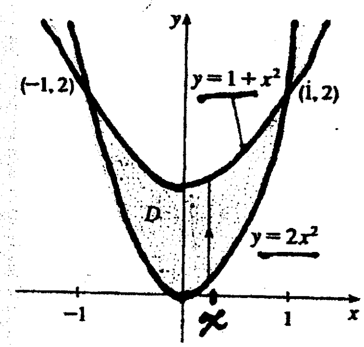


FIGURE 8

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} dx$$

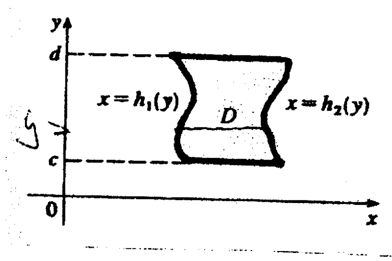
$$= \int_{-1}^{1} \left[x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2\right] dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\Big]_{-1}^{1} = \frac{32}{15}$$

$$D = \{(x, y) \mid -1 \le x \le 1,\ 2x^2 \le y \le 1 + x^2\}$$

Type II Regions



$$c \le y \le d$$
$$h_1(y) \le x \le h_2(y)$$

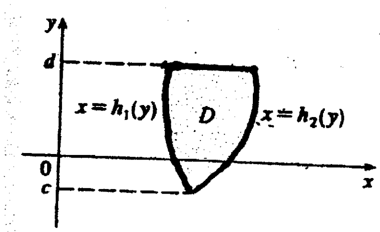
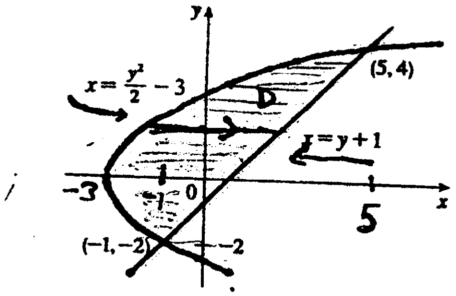


FIGURE 7
Some type II regions

> **5** $$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$
>
> where D is a type II region given by Equation 4.

A Type II Region Example

D is the region bounded by the graphs of $x=\frac{y^2}{2}-3$ and $x=y+1$.



(b) D as a type II region

FIGURE 12

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy = \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

$$= \frac{1}{2}\int_{-2}^{4} y\left[(y+1)^2-\left(\tfrac{1}{2}y^2-3\right)^2\right] dy$$

$$= \frac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4}+4y^3+2y^2-8y\right) dy$$

$$= \frac{1}{2}\left[-\frac{y^6}{24}+y^4+2\frac{y^3}{3}-4y^2\right]_{-2}^{4} = 36$$

$$D=\left\{(x,y)\;\middle|\;-2\le y\le 4,\ \left(\tfrac{1}{2}y^2-3\right)\le x\le (y+1)\right\}$$

The Same Region Viewed As A Type I Region

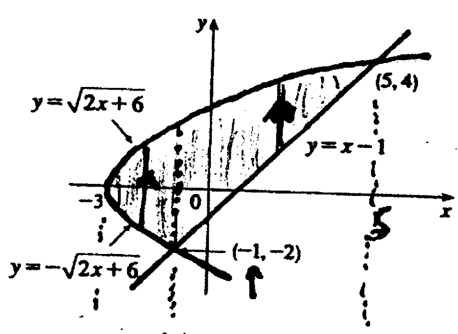


FIGURE 12

$-3\le x\le -1$ $\quad$ $-1\le x\le 5$

$x=\frac{y^2}{2}-3$	$x=y+1$
$2x=y^2-6$	$x-1=y$
$2x+6=y^2$	$y=x-1$
$y=\pm\sqrt{2x+6}$	

$$\iint_D xy\,dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

EVALUATION REQUIRES TWO ITERATED INTEGRALS!

EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$.

$$\underbrace{\int_0^1 \int_{y=x}^{y=1} \sin(y^2)\,dy\,dx}$$

THIS WOULD EVALUATE

$$\iint_D \sin(y^2)\,dA,$$

but $\int \sin(y^2)\,dy$ requires an infinite power series representation.

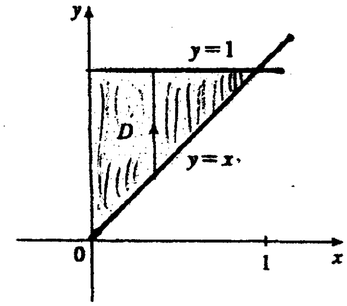


FIGURE 15
D as a type I region

$D = \{(x, y) \mid 0 \leq x \leq 1,\ x \leq y \leq 1\}$

REVERSING THE ORDER OF INTEGRATION CAN SOMETIMES RESULT IN AN ITERATED INTEGRAL WHICH IS EASIER TO EVALUATE.

SKETCH THE REGION AND RE-INTERPRET THE REGION AS A REGION OF THE OTHER TYPE.

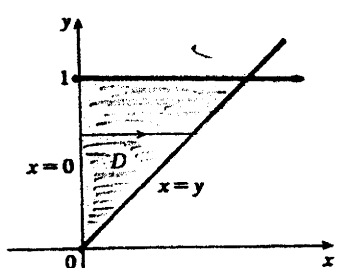


FIGURE 16
D as a type II region

$D = \{(x, y) \mid 0 \leq y \leq 1,\ 0 \leq x \leq y\}$

HERE, THE SAME REGION IS VIEWED AS A TYPE II REGION.

$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \iint_D \sin(y^2)\,dA$$

$$= \int_0^1 \int_0^y \sin(y^2)\,dx\,dy = \int_0^1 \Big[x\sin(y^2)\Big]_{x=0}^{x=y}\,dy$$

$$= \int_0^1 y\sin(y^2)\,dy = -\tfrac{1}{2}\cos(y^2)\Big]_0^1$$

$$= \tfrac{1}{2}(1 - \cos 1) \approx 0.22985$$